

Chaotic control and synchronization for system identification

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Research into applications of synchronized chaotic systems assumes that it will be necessary to build many different drive-response pairs, but little is known in general about designing higher dimensional chaotic flows. In this paper, I do not add any design techniques, but I show that it is possible to create multiple drive-response pairs from one chaotic system by applying chaos control techniques to the drive and response systems. If one can design one chaotic system with the desired properties, then many drive-response pairs can be built from this system, so that it is not necessary to solve the design problem more than once. I show both numerical simulations and experimental work with chaotic circuits. I also test the response systems for ability to overcome noise or other interference.

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I. INTRODUCTION

There has been much work on using synchronized chaos [1–13] for applications such as communications or radar, but a constant assumption in all of this work has been that designing many different chaotic systems (for different transmitters, for example) would not be difficult. Simply designing new chaotic systems has not proven too difficult, as Sprott has shown [14], but designing chaotic flows with specific properties in mind is considerably more difficult. There are design techniques for one dimensional maps, and several authors [15,16] have shown how to extend these techniques to a continuous system, but these techniques are suitable only for a very limited range of chaotic systems. If one wants a chaotic flow that may be built as a high-frequency circuit, for example, the behavior of the available circuit elements is complex enough that the circuit design must proceed experimentally.

I do not attempt to solve the general design problem, but instead I show that if one chaotic system with the desired properties may be built, then control techniques, such as the OGY technique [17,18], may be used to create multiple drive-response pairs from the desired chaotic system.

Instead of designing a response system to synchronize to a particular chaotic drive system, the response system will synchronize to a particular chaotic trajectory. Chaotic control techniques are used to select the drive trajectory, and control techniques are also used to make the response system synchronize or not synchronize to the drive system.

The basic method is the following.

(1) Allow the chaotic drive system to follow a chaotic trajectory of finite length L , and store control information about this trajectory.

(2) Control the drive system to always follow this finite length chaotic trajectory. Since the trajectory is finite length, it must be repeated, so the system is actually periodic with period L .

(3) While the drive system follows a designated trajec-

tory, use a signal from the drive system to drive a response system. The response system need not be identical to the drive system, as generalized synchronization can be useful for some applications.

(4) While this response system is being driven, store control information about it.

(5) Use the stored control information to control the driven response system. The response system trajectory will be the same as it was without control if the same drive signal is being used.

(6) If the drive signal is now switched to a different signal, then the response system trajectory will be different.

The drive system can be controlled to follow different trajectories. Each different drive system trajectory has a corresponding response system trajectory when the response is uncontrolled. Using the matching response control sequence will not alter the response trajectory, but using a nonmatching response control sequence will alter the response from the uncontrolled trajectory.

II. CHAOTIC SYNCHRONIZATION

I assume a chaotic drive system of the form

$$\dot{\vec{x}} = f(\vec{x}), \quad (1)$$

and a response system of the form

$$\dot{\vec{y}} = g(\vec{y}) + h(\vec{x}), \quad (2)$$

where \vec{x} and \vec{y} are vectors, and $h(\vec{x})$ is a function of \vec{x} . The coupling in Eq. (2) is a linear coupling, as are all types of coupling used in this paper, but other types of coupling are also possible.

For synchronization to occur, all the Lyapunov exponents of the dynamical system of Eq. (2) must be negative. If the function g is identical to the function f , then identical synchronization is possible, otherwise the synchronization is said to be generalized synchronization. Detecting identical synchronization is easy, as signals in the response system will be almost the same (within experimental limits) as sig-

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nals in the drive system. Detecting generalized synchronization is more difficult and, in fact, there are many different definitions for generalized synchronization [12]. For this paper, we choose a response system with only one basin of attraction, and for synchronization we require only that the response system have all negative Lyapunov exponents. In this case, one may use an auxiliary system to detect synchronization [16]: two copies of the response system are built, and their outputs are compared to each other. If the outputs of the two response systems match (within experimental error), then the response is said to be synchronized to the drive.

III. CONTROL

Ott, Grebogi, and Yorke (OGY) [17] showed that only small perturbations were necessary to control a chaotic system if the control kept the system near solutions of its equations of motion, such as unstable periodic orbits. Hayes *et al.* [19] later showed that one could encode information by using OGY control to switch between different trajectories of a chaotic system. As Hayes pointed out, the availability of multiple trajectories was a consequence of the positive entropy of a chaotic system. Hayes felt that this positive entropy should make chaotic signals natural information carriers. Hayes and others [11,19–21] have shown that one may use different states of the chaotic system as symbols, and control may be used to determine which symbol sequences are transmitted.

In this work, chaos control techniques are used to generate multiple different trajectories from a chaotic system. These trajectories have a finite length, so they must be repeated, but the trajectories may be chosen long enough that they still have broad band spectra. A chaotic response system is also controlled so that it will synchronize only to one trajectory from a chaotic transmitter, and not to any others.

IV. NUMERICAL WORK

I first demonstrate control and synchronization in a numerical experiment. In this numerical example, the drive and response systems are identical, so identical synchronization is seen. The chaotic system here is three dimensional, and I measure the length of a chaotic trajectory by the number of times that the variable x_2 crosses 0 in the positive direction.

The drive system is similar to the piecewise-linear Rossler system [22]:

$$\begin{aligned} \frac{dx_1}{dt} &= -\alpha(0.05x_1 + 0.5x_2 + x_3), \\ \frac{dx_2}{dt} &= -\alpha(-x_1 - 0.3x_2), \\ \frac{dx_3}{dt} &= -\alpha(-g(x_1) + x_3), \end{aligned} \quad (3)$$

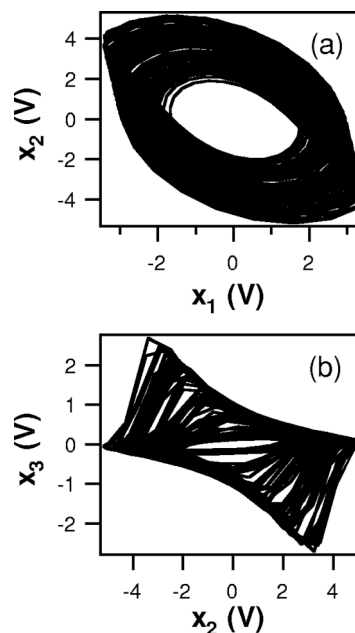


FIG. 1. Attractors for the chaotic system of Eq. (3).

$$g(x) = \begin{cases} m_1x + b_2 & x \leq -x_0 \\ m_0x, & -x_0 < x < x_0 \\ m_1x + b_1, & x \geq x_0 \end{cases}$$

where $\alpha=10.0$, $m_0=0.1$, $m_2=15.0$, $x_0=3.0$, $b_1=x_0(m_0-m_1)$, and $b_2=-b_1$. Figure 1(a) is a plot of x_2 versus x_1 , while Fig. 1(b) is a plot of x_3 versus x_2 . These equations were integrated with a fourth order Runge-Kutta integrator with a time step of 0.04 s.

The response system is a duplicate of the drive system. The response system is described by

$$\begin{aligned} \frac{dy_1}{dt} &= -\alpha(0.05y_1 + 0.5y_2 + y_3), \\ \frac{dy_2}{dt} &= -\alpha[-y_1 - 0.3y_2 + c(y_2 - x_2)], \\ \frac{dy_3}{dt} &= -\alpha[-g(y_1) + y_3], \\ \frac{dz}{dt} &= \alpha(|x_2 - y_2| - z), \end{aligned} \quad (4)$$

where the variable z is a measure of the average synchronization error and the coupling constant $c=0.1$.

An outline of the experiment is the following.

(1) Allow the drive system to evolve freely for 100 cycles (measured by the x_2 0 crossing), after allowing for initial transients to die down.

(2) Each time x_2 crosses 0 in the positive direction, record the values of x_1 and x_3 . The sequence of 100 x_1 and x_3 values is the control sequence for the drive. Call this control sequence chaos 1.

(3) Repeat for different initial conditions to get the control sequence chaos 2.

(4) Control the drive system with chaos 1 by waiting for x_2 to cross 0 in the positive direction and then setting x_1 and x_3 equal to their corresponding values from the control sequence chaos 1. Use the x_2 signal from the drive system to drive the identical response system [Eq. (4)]. The control sequences have a finite length, so the controlled system is now periodic, but with a period of 100.

(5) The response system is now being driven by a controlled signal from the drive system (which is controlled by chaos 1), and it will synchronize exactly to the drive system. When the response variable y_2 crosses 0 in the positive direction, record the values of y_1 and y_3 to get the response control sequence response 1. When the drive system is controlled by chaos 2, repeat the same procedure to get the response control sequence response 2.

A. Phase Control

When the uncontrolled response system of Eq. (4) is driven by the uncontrolled drive system of Eq. (3), identical synchronization occurs; that is, y_1 , y_2 and y_3 approach x_1 , x_2 and x_3 . If drive and response are controlled by the same control sequence, identical synchronization will also occur, but only if the drive and response control sequences are in phase with each other.

The variable z in Eq. (4) is used to help judge if the drive and response control sequences are in phase. With control on for the response system, when y_2 crosses 0 in the positive direction, the value of z is compared to some threshold. If z is less than the threshold, then it is assumed that drive and response are synchronized, and y_1 and y_3 are set to the appropriate values in the control sequence. If z exceeds the threshold, then it is assumed that the drive and response control sequences are out of phase, and so the phase of the response control sequence is advanced by 1 before control is applied. In a manner similar to digital code division multiple access (CDMA) [23], the response control sequence is advanced at a faster rate than the drive control sequence until synchronization is obtained.

Figure 2 shows the results of the control when a threshold of $z=0.2$ is used. Figure 2 is a plot of $\delta=x_1-y_1$. In Fig. 2(a), the response system is free running ($c=0$) for the first 40 s. At $t=40$ s, c is set to 0.1 and control is applied to the response. Initially the drive and response control sequences are out of phase, so there is an initial transient (which should depend on the length of the control sequence) before synchronization is obtained at about 200 s. In Fig. 2(b), different control sequences are used for drive and response. Once again, control is started at 40 s, but because drive and response control sequences are different, no synchronization is seen.

The utility of this signal recognition method will depend on how many different control sequences can be produced. It should be possible to estimate the number of different control sequences by assigning symbol sequences to the control sequences. First, a generating partition for the chaotic system must be found. While this is a difficult process in general,

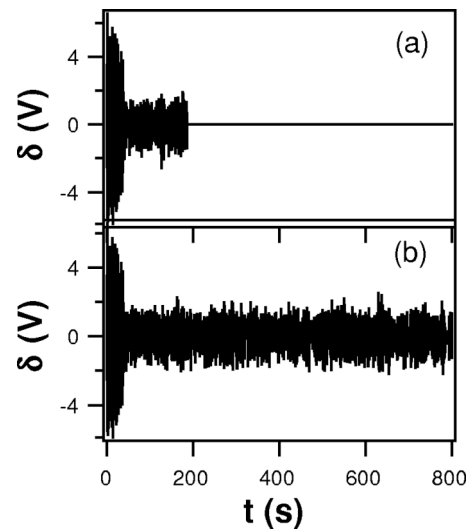


FIG. 2. Results of driving and controlling the response system of Eq. (4). $\delta=x_1-y_1$ is the difference between drive and response systems. In (a), the proper response control sequence was used, resulting in synchronization after a transient. In (b), the wrong response control sequence was used, resulting in no synchronization.

there are some recent methods for finding such partitions [24,25]. The generating partition is then used to define a set of symbols; i.e., if the chaotic trajectory passes through one region, one particular symbol is produced, while if it passes through a different region, a different symbol may be produced. It may be that the chaotic system possess a grammar, so that not all symbol sequences occur. The number of different symbol sequences that occur for a given sequence length L should correspond to the number of possible control sequences.

This simple flow is useful to show how the control and synchronization method works, but it is not practical. Adding Gaussian white noise with an amplitude of 0.02 or greater destroys synchronization. Below, I show a circuit example which is more robust to additive noise.

V. CIRCUIT EXPERIMENTS

The circuit used for these experiments is based on a chaotic system that maintains phase synchronization even when noise much larger than the transmitted signal is present [26,27]. This system consists of a Rossler-like chaotic circuit which operates in one frequency range coupled to a stable (nonoscillating) system which operates in a much lower frequency range. The separation of frequencies allows the lower-frequency part of the response circuit to stay in phase synchronization to the lower-frequency part of the drive system. This is a fairly complicated chaotic circuit, and designing a circuit with these noise-robust properties was difficult, so it is desirable to produce many drive-response pairs from this circuit, rather than having to design more circuits with the same properties.

The circuits used were built using operational amplifiers. The drive circuit may be approximately described by the equations

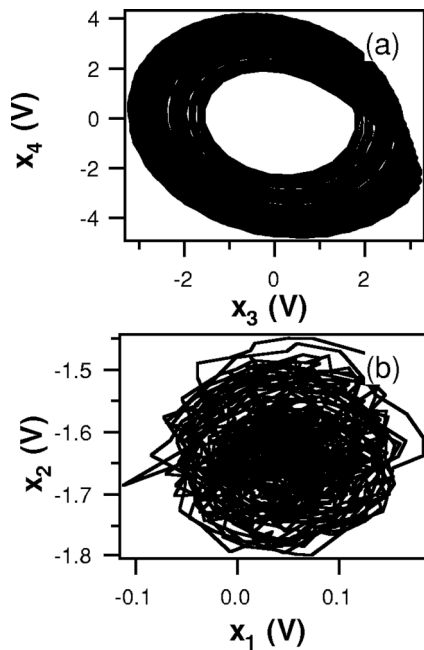


FIG. 3. Attractors for the chaotic circuit used to provide a driving signal. (a) is the attractor from the fast part of the circuit, while (b) is the attractor for the slow part of the circuit.

$$\begin{aligned}
 \frac{dx_1}{dt} &= -\frac{1}{RC_1}(0.02x_1 + 0.5x_2 + 0.5|x_4|), \\
 \frac{dx_2}{dt} &= -\frac{1}{RC_1}(-x_1 + 0.02x_2), \\
 \frac{dx_3}{dt} &= -\frac{f(x_1)}{RC_2}(0.02x_3 + 0.5x_4 + x_5 + 0.1x_1), \\
 \frac{dx_4}{dt} &= -\frac{f(x_1)}{RC_2}(-x_3 - 0.13x_4), \\
 \frac{dx_5}{dt} &= -\frac{f(x_1)}{RC_2}[-g(x_3) + x_5], \\
 g(x) &= \begin{cases} 0, & x < 3 \\ 15(x-3), & x \geq 3 \end{cases}, \\
 f(x) &= 1 + 0.2(x + 1.75),
 \end{aligned}
 \tag{5}$$

where $R=100\text{ k}\Omega$, $C_1=0.1\text{ }\mu\text{F}$, and $C_2=0.001\text{ }\mu\text{F}$. For these parameters, the signal x_1 has a frequency of approximately 10.5 Hz, while x_3 has a frequency of about 946 Hz. Figure 3(a) is a plot of x_2 versus x_1 , and Fig. 3(b) is a plot of x_4 versus x_3 . The function $f(x)$ serves to broaden the spectrum of the fast signals (x_3 through x_5).

The signal that is actually transmitted is x_t , defined by

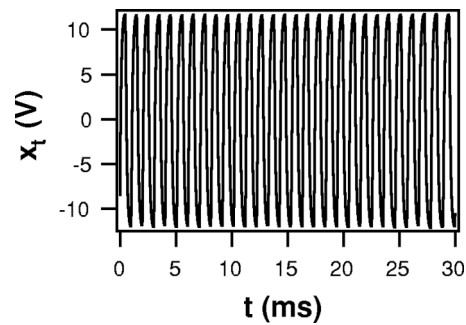


FIG. 4. Transmitted signal x_t produced by the chaotic driving circuit.

$$\frac{dx_t}{dt} = -\frac{1}{RC_2} \left[sq \left(\frac{x_4}{x_3^2 + x_4^2} \right) + x_t \right], \tag{6}$$

where the $sq(x)$ function means that $sq(x)=15\text{ V}$ if $x>0$ and $sq(x)=-15\text{ V}$ if $x<0$. The $sq(x)$ function was executed by an operational amplifier with a very large gain. The integrator was used as a low pass filter so that x_t was not a square wave.

Figure 4 is a plot of x_t as a function of time, while Fig. 5 is its power spectrum. The signal x_t has a constant envelope, which makes it more efficient to transmit, and makes it easier to restore its amplitude to a known value after transmission.

The response circuit may be described by the equations

$$\begin{aligned}
 \frac{dy_1}{dt} &= -\frac{1}{RC_1}(0.1y_1 + 0.5y_2 + 0.5|y_3|), \\
 \frac{dy_2}{dt} &= -\frac{1}{RC_1}(-y_1 + 0.1y_2), \\
 \frac{dy_3}{dt} &= -\frac{1}{RC_2}(0.02y_3 + 0.5y_4 + 0.1y_1), \\
 \frac{dy_4}{dt} &= -\frac{1}{RC_2}(-y_3 - ky_t),
 \end{aligned}
 \tag{7}$$

where R , C_1 , and C_2 are the same as in Eq. (5). The constant k is used to alter the amplitude of the transmitted signal x_t .

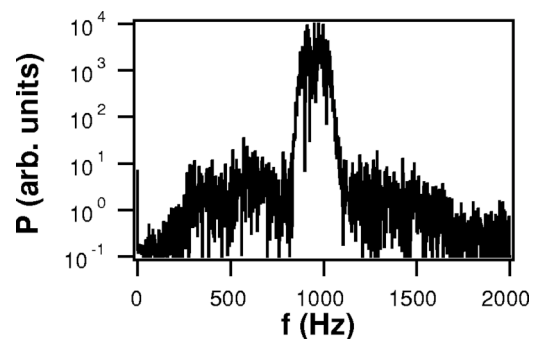


FIG. 5. Power spectrum of the signal x_t produced by the chaotic drive circuit.

The response circuit does not match the drive circuit, which means that exact synchronization is not possible. In order to determine when generalized synchronization took place, the auxiliary system approach was used [28]. A second response circuit that was identical (within experimental error) was built. In order to improve the matching between circuits, resistors with a 1% tolerance were used, and a 20 turn potentiometer was used in the integrator for the y_1 signal to correct the time constant $1/RC_1$ for error in the capacitor value. The y_1 signals from the two response circuits were compared to determine if generalized synchronization was occurring.

The control methods used for the circuit were similar in principle to those used for the numerical experiment, but some details were different. Rather than try to control the drive circuit as in the numerical section, a 10 000 point signal x_t from the drive circuit was digitized at 20 000 points/s and played back through an arbitrary wave form generator. The playback rate was chosen so that the frequency of the signal from the arbitrary wave form generator matched the frequency of the original drive signal. Chaotic signals were recorded at two different times, resulting in two different chaotic sequences, labeled as chaos 1 and chaos 2. The chaotic signals were played back with a peak to peak amplitude of 1.98 V, and the drive constant k in Eq. (7) was set to 1.0.

For the control of the response circuit, the y_1 signal was first passed through a 1 μ F capacitor to remove the dc component. This signal was then integrated by an operational amplifier integrator to smooth out any residual ripple in y_1 , producing the signal ψ :

$$\frac{d\psi}{dt} = -\frac{1}{RC_1}(y_1 + 0.1\psi), \quad (8)$$

where R and C_1 were previously defined. Several logic circuits were then used to give a short +5 V pulse when ψ crossed 0 in the negative direction.

In order to record the necessary control information, the response circuits were driven by the recorded x_t signal from the drive circuit, which had been controlled by the sequences chaos 1 or chaos 2. When ψ crossed 0 in the negative direction, the value of y_1 was stored for the response control sequence. The response control sequence when the drive circuit was controlled by chaos 1 was response 1, and when the drive was controlled by chaos 2, the response control sequence was response 2.

During control, the response circuits were driven by the recorded x_t signal from the drive circuit, which had been controlled by the sequence chaos 1 or chaos 2. When ψ crossed 0 in the negative direction, the difference between y_1 and the corresponding signal from the matching auxiliary circuit y_{1a} was compared to a fixed threshold in the computer. If $|y_1 - y_{1a}| > 0.3$, it was assumed that the response circuits were not synchronized, and the phase of the response control sequence was advanced by 1. If the difference was less than the threshold, the control phase was not advanced. For either result, the computer then set y_1 for the circuit to the next value in the response control sequence, after which the response control sequence phase was advanced. The se-

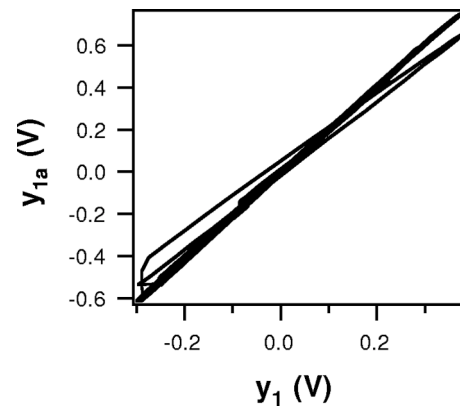


FIG. 6. Plot showing synchronization of the response circuit (y_1) and the auxiliary response circuit (y_{1a}), confirming generalized synchronization when the correct response control sequence for a particular drive signal is used.

quences chaos 1 and chaos 2 corresponded to five cycles of the slow part of the circuit, so each control sequence had a length of 5.

Figure 6 is a plot of y_{1a} versus y_1 when the arbitrary wave form generator is playing back the drive signal x_t from a drive circuit controlled by chaos 1 and the response circuit is being controlled by the control sequence response 1. There are some occasional small departures from synchronization, but most of the time the two auxiliary systems are synchronized. Figure 7 is the same plot when the drive circuit was controlled by chaos 2 but the response control sequence was still response 1. There is a definite loss of synchronization, so the pair of response circuits are able to recognize the difference between chaos 1 and chaos 2.

This circuit can still recognize the difference between chaos 1 and chaos 2 when noise is present. The arbitrary wave form generator was used to produce a Gaussian white noise signal with a bandwidth of 50 kHz, which was added to the drive signal x_t from a drive circuit controlled by chaos 1. Figure 8 shows y_{1a} versus y_1 when the drive circuit was controlled by chaos 1 and noise was added to x_t , with a signal power to noise power ratio of 0.7(-1.4 dB). The re-

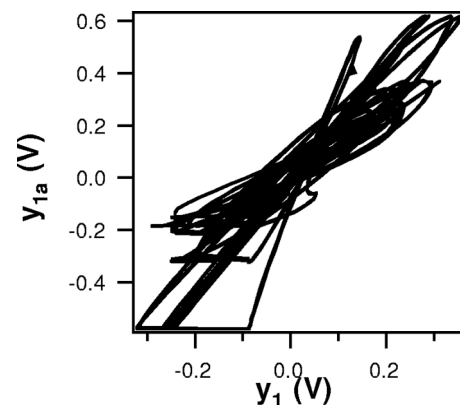


FIG. 7. Plot showing a lack of generalized synchronization between the response circuit y_1 and the auxiliary response circuit y_{1a} when a response control sequence that does not correspond to the drive signal is used.

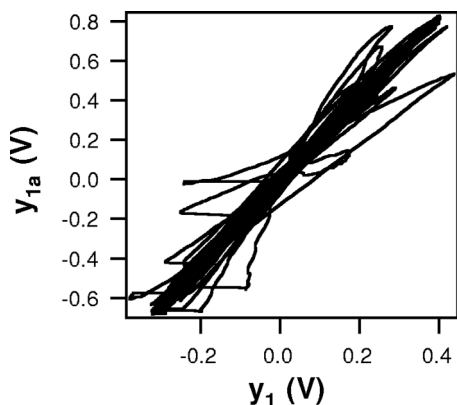


FIG. 8. Plot of the auxiliary response circuit y_{1a} vs the response circuit y_1 showing that generalized synchronization is maintained even when additive Gaussian white noise larger than the drive signal is present.

response system was controlled by the control sequence response 1. The synchronization is still recognizable when Fig. 8 is compared to Fig. 7, where there was no noise, but the wrong control sequence was used. The cross correlation at 0 time lag between y_1 and y_{1a} when the wrong control sequence was used but no noise was present was 0.93, while the cross correlation when the correct control was used but the signal to noise ratio was 0.7 was 0.98.

The effect of interference from another chaotic signal on the response circuits was also tested. A second arbitrary wave form generator was used to play back the transmitted signal from a drive circuit controlled by chaos 2. This second transmitted signal x_{i2} was added to the x_i signal from a drive circuit controlled by chaos 1. When both x_i and x_{i2} had the same amplitude, the cross correlation between y_1 and y_{1a} was 0.96. When the x_{i2} signal amplitude was 1.5 times the amplitude of the x_i signal, the cross correlation dropped to 0.91, lower than the value when the wrong drive signal was used. The response circuits can reject some interference, but they

have trouble if the interference is too similar to the driving signal.

VI. CONCLUSIONS

Reference [11] also uses chaos control to synchronize two chaotic systems, but the content of that paper is very different from this work. In Ref. [11], the symbol sequence for the chaotic trajectory is not known at the receiver, but is communicated through the channel. Reference [11] uses the properties of these symbol sequences to determine the maximum possible precision of synchronization for identical, lag, or anticipated synchronization.

In the present work, the symbol sequence corresponding to the drive system trajectory is already known at the receiver. This information is used to determine which trajectory is being sent. The technique in this paper is similar to CDMA [23], where different orthogonal sequences are used to identify different transmitters.

The control and synchronization procedure should make it easier to design multiple drive-response pairs, as it is not necessary to build a completely different chaotic circuit for each pair. Since the spectra of the different drive systems are the same (for a long enough trajectory), it should also be possible to make better use of frequency space by using the same frequency band for many different drive-response pairs.

In the circuit experiments, the ability of a controlled response system to recognize a particular signal in the presence of noise or interference was tested. It has been shown in previous work that the noise robustness of similar two frequency circuits may be improved by increasing the separation between fast and slow frequencies [26]. Resistance to chaotic interference was not as good, but designing the chaotic drive system so that different output sequences were less similar to each other should increase the resistance to this type of interference. The control techniques used here also allow greater freedom in designing the transmitter, since the receiver no longer has to be a replica of the transmitter.

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